

Global exponential Stabilization of Language Constrained Switched System based on the S-procedure approach

Yang Song, *Member, IEEE*, Yunyun Jin, Wang Yan, Taicheng Yang

Abstract— This paper considers global exponential stabilization (GES) of switched system under language constraint which is generated by a non-deterministic finite state automaton. The S-procedure characterization is employed to provide sufficient conditions of GES which are less conservative than the existing Lyapunov-Metzler condition. Moreover, by revising the construction of Lyapunov matrices and the min-switching control policy, a more flexible result is obtained such that stabilization path at each moment might be multiple. Finally, a numerical example is given to illustrate the effectiveness of the proposed results.

I. INTRODUCTION

Switched system is a class of hybrid systems which consists of a set of modes and a switching law orchestrating the switching between multiple modes. The study on switched system has attracted much attention for their capability of modeling complex practical control systems, such as manufacturing systems [1], distributed communication [2], and networked control system [3], etc. Stability analysis of switched system is a fundamental problem, the methods for which are closely related to switching sequence [4]. Generally, the switching sequence can be divided into two categories: arbitrary switching and constrained switching.

For the stability under arbitrary switching, the most direct method is to demonstrate the existence of a common Lyapunov function (CLF). However, CLF is only a sufficient condition and is usually difficult to be found [5]. In [6], a necessary and sufficient condition is presented based on the joint spectral radius (JSR), i.e. a switched system is stable if and only if the JSR is less than one. In general, the computation of JSR is a challenging task. A method called path-complete graph Lyapunov function (GLF) is introduced to estimate JSR [7][8]. Moreover, [9] presents a method for

constructing CLF based on path-complete GLF. By extending the concept of GLF, the approach of graph control Lyapunov function (GCLF) is proposed to investigate the exponential stability of switched system [10], where a weighted digraph is constructed to represent the multiple Lyapunov inequalities.

Constrained switching is generally classified as time-driven, state-driven. Many stability results have been proposed for the time-driven switched system in which the constraint is generally imposed on the dwell-time or the average dwell-time of switching sequence [11][12]. For the state-driven switching, the stability can be judged by a set of so-called Lyapunov-Metzler (L-M) inequalities [13][14]. This Lyapunov-Metzler condition is further relaxed by using the technique of S-procedure [14]. Recently, much attention has been paid on the switched system with formal language constraint which are often described in form of automaton (see [15][16][17] for details). This language constrained switching exists extensively in many practical systems. For example, in car shift control, the switching from the first gear to second gear is allowable, but the shift from the first gear to fourth gear is forbidden. In a broad sense, language constraints can also be regarded as state-driven switching since the switching depend on the states and the edges of the automaton.

Stability analysis of language constrained switched system (LCSS) is an interesting topic recently. Literature [18] defined the constrained joint spectral radius (CJSR), and proved that a LCSS to be stable is equivalent to the CJSR is less than one. Several approaches for estimating CJSR with arbitrarily accurate approximation are given in [19][20]. Furthermore, some new definitions on the stability for LCSS with deterministic finite state automaton constraints, such as absolute asymptotic stability, shuffle asymptotic stability, etc., are given and their inherent relations are analyzed [21]. For the LCSS with the constraint generated by a nondeterministic finite state automaton (NFA), a new concept of recurrent stabilizability and the corresponding necessary and sufficient condition are proposed [22]. Based on L-M inequalities and by combining the min-switching control policy, a sufficient condition for the stabilizability of LCSS is presented [23].

This paper deals with the stabilization of language constrained switching system. Inspired by [24], we proposed new sufficient conditions of global exponential stabilizability for LCSS, which are less conservative than the L-M condition in [23]. It should be noted that, in the previous results [19]-[23], a set of Lyapunov matrices are constructed in which each Lyapunov matrix corresponds to a vertex of the

*Research supported by the National Natural Science Funds of China (61573237) and 111 Project (D18003).

Yang Song is with the School of Mechatronic Engineering and Automation, Shanghai University, and Shanghai Key Laboratory of Power Station Automation Technology, Shanghai, 200072, China (e-mail: y_song@shu.edu.cn).

Yunyun Jin is with the School of Mechatronic Engineering and Automation, Shanghai University, Shanghai, 200072, China (e-mail: 177220241@shu.edu.cn).

Wang Yan is with the Engineering Center of IoT Application Technology (Ministry of Education) and the School of Internet of Things Engineering, Jiangnan University, Wuxi 214122, China (e-mail: wangyan@jiangnan.edu.cn).

Taicheng Yang is with the Department of Engineering and Design, University of Sussex, Brighton, BN1 9QT, UK (e-mail: t.c.yang@sussex.ac.uk).

automaton respectively. To loosen the conservativeness, this paper introduces two set of independent matrices in the proposed results, i.e. the set of Lyapunov matrices as in [23], and another set of auxiliary matrices which corresponds to the vertexes or the edges of the automaton. Furthermore, we show that if there exist multiple sets of feasible solutions for the auxiliary matrices, then extra flexibility on the selection of the switching path for stabilization can be acquired by applying the improved the min-switching control policy.

The rest of this paper is organized as follows. The problem formulation, main results, a numerical example and conclusions are presented in sections 2, 3, 4 and 5 respectively.

Notations: \mathbb{N} is the set of natural numbers, $\mathbb{N}_n = \{x \in \mathbb{N} : 1 \leq x \leq n, n \in \mathbb{N}\}$, $\mathbb{R}^{n \times n}$ is the space of $n \times n$ matrices with real entries. $|\Sigma|$ is the number of the elements in the set Σ . $(\Pi)_{i,j}$ denotes the element of matrix Π in the i -th row and the j -th column. A matrix \mathcal{M} is called rectangular Metzler matrix if each element is nonnegative and the sum of each column is unitary.

II. PROBLEM FORMULATION

Consider the following L-SLS with q modes,

$$\dot{x}(t+1) = A_{\sigma(t)} x(t) \quad (1)$$

where system state $x(t) \in \mathbb{R}^n$ and the switching law $\sigma(\cdot) : \mathbb{R} \rightarrow \mathbb{N}_q$, which is subject to the constraint given in the form of a non-deterministic finite state automaton \mathcal{A} , i.e., each switching sequence must be an element of the language generated by \mathcal{A} .

Definition 1. (Non-deterministic finite automaton, NFA). A non-deterministic finite automaton is a tuple $\mathcal{A} = (S, \Sigma, \delta, S_0, F)$, where S is the set of the automaton states, Σ denotes the labels set, δ is the set of set-valued transition map $\delta : S \times \Sigma \rightarrow 2^S$, S_0 is the set of initial states with $S_0 \subset S$, F is the set of acceptable states with $F \subset 2^S$.

In this paper, a transition sequence of \mathcal{A} is denoted by $\Gamma : (s_1, \sigma_1, s_2) (s_2, \sigma_2, s_3) \dots (s_i, \sigma_i, s_{i+1}) \dots$, where $s_i, s_{i+1} \in S$, $\sigma_i \in \Sigma$, $s_{i+1} \in \delta(s_i, \sigma_i)$, $\forall i \in \mathbb{N}$. The transition sequence Γ is said to be acceptable if there exists a positive integer N such that for all $i \geq N$, $s_i \in F$. Each label of the automaton corresponds to a mode of the system, which means that the label set equals to the system-mode set, i.e. $\Sigma = \mathbb{N}_q$.

Definition 2. (Labeled digraph). A labeled digraph is denoted by $G_{\mathcal{A}}(V, E)$, where V, Σ are the set of vertices and labels respectively, $E \subseteq V \times \Sigma \times V$ is the set of labeled edges. For a labeled edge $e = (u, l, v)$, the three elements u, l, v are denoted by $\text{tail}(e)$, $l(e)$ and $\text{head}(e)$ respectively. A self-loop is a labeled edge with $\text{tail}(e) = \text{head}(e)$. If a sequence of labeled edges $p : e_1 e_2 \dots e_i \dots$ satisfies $\text{tail}(e_{i+1}) = \text{head}(e_i)$, $\forall i \in \mathbb{N}$, then the sequence p is called a path of the labeled digraph. The length of p , i.e. the number of the labeled edges in the path, is denoted by $\text{len}(p)$. Figure 1 is an example of $G_{\mathcal{A}}(V, E)$. In

this figure, $e_1 = (v_1, 1, v_2)$, where $\text{tail}(e_1) = v_1$, $l(e_1) = 1$, $\text{head}(e_1) = v_2$, and $e_2 = (v_2, 3, v_2)$ is a self-loop.

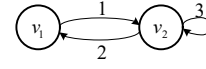


Figure 1. An example of a simple labeled digraph

An NFA \mathcal{A} can be represented by a labeled digraph and denoted as $G_{\mathcal{A}}(V, E)$, where $V = S$, a labeled edge $e = (u, l, v)$ corresponding to a transition (s_i, σ_i, s_{i+1}) in automaton \mathcal{A} , therefore a path p corresponds to a transition sequence Γ . Let $P_M(\mathcal{A})$ (resp. $P_{\infty}(\mathcal{A})$) denotes the set of acceptable paths with length M (resp. infinite length). All the label sequences $l(p) = l(e_1)l(e_2) \dots l(e_i) \dots$ of acceptable paths constitute the language generated by \mathcal{A} , which is denoted as $L(\mathcal{A})$.

Definition 3. (Strongly connected component, SCC [25]). The labeled digraph $G_{\mathcal{A}}(V, E)$ is strongly connected if there exists a finite path from any vertex u to any other vertex v , $\forall u, v \in V$, especially, a labeled digraph $G_{\mathcal{A}}(V, E)$ with only one vertex is also regarded as strongly connected. A subgraph $\Omega(V', E')$ of $G_{\mathcal{A}}(V, E)$ is a strongly connected component (SCC) if $\Omega(V', E')$ is strongly connected and any other subgraph of $G_{\mathcal{A}}(V, E)$ strictly containing $\Omega(V', E')$ is not strongly connected. A SCC is called trivial if there is only one vertex and no self-loops, otherwise it is nontrivial.

Definition 3 implies the following two facts: 1) a labeled digraph $G_{\mathcal{A}}(V, E)$ can be divided into one or several SCCs; and 2) for each SCC, one can always find an infinite path which ultimately enters and remains in such SCC after finite steps.

A method called Dijkstra's algorithm [27] can be applied to determine which nontrivial SCC is entered in a given step. This paper investigates whether an L-SLS is globally exponentially stabilizable after the path enters the SCC.

Definition 4. (Globally exponentially stabilizable, GES). L-SLS (1) is GES in a nontrivial SCC $\Omega(V', E')$ if for any $x(0) \in \mathbb{R}^n$, there exists a scalar T and a path p generated by $G_{\mathcal{A}}(V, E)$ such that the trajectory of (1) under the path p satisfying

$$\|x_{x(T)}^p(t)\| \leq ce^{-\mu(t-T)} \|x(T)\|, \quad \forall t \geq T$$

where $c > 0$ and the decay rate $\mu > 0$, $x_{x(T)}^p(t)$ denotes the system state at time t which starts from the state $x(T)$ and evolves under path p .

III. MAIN RESULTS

In this section, a lift-technique is applied to reconstruct L-SLS (1) into the M-step system (2) to give a more flexible sufficient condition of GES.

$$\hat{x}(t+1) = \hat{A}_{\hat{\sigma}(t)} \hat{x}(t) \quad (2)$$

where $\hat{A}_{\sigma(t)} \in A_{\Sigma_M} = \{A_1 \cdots A_M \mid l_1 \cdots l_M \in \Sigma_M\}$, the switching law $\hat{\sigma}(\cdot): \mathbb{R} \rightarrow \mathbb{N}_{|\Sigma_M|}$. The language constraint $G_{\mathcal{A}}^M(V, \hat{E})$ is lifted from $G_{\mathcal{A}}(V, E)$, where each path with length M in $G_{\mathcal{A}}(V, E)$ is replaced by a corresponding edge with label $l_p = l_1 l_2 \cdots l_M$, as shown in Figure 2(a) and Figure 2(b). The set of all new labels l_p is denoted by Σ_M . Besides, if the L-SLS is also subject to dwell time constraints, then the dwell time constraints can be reflected intuitively on such M-lifted labeled digraph $G_{\mathcal{A}}^M(V, \hat{E})$. For example, if the dwell time $\tau = 3$ for each mode of L-SLS (1), then the overall constraints on switching can be further reduced as the labeled digraph Figure 2(c).

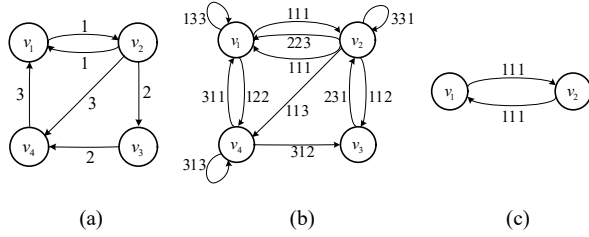


Figure 2. A labeled digraph (a) and M-lifted labeled digraph (M=3 for (b); M=3 with $\tau = 3$ for (c)).

Lemma 1: The M-step L-SLS (2) is GES if and only if L-SLS (1) is GES.

Proof: The proof here is similar to the proof in [26], and thus it is omitted here. ■

Thus, the GES problems of L-SLS (1) can be investigated through by the lifted L-SLS (2). Let $(\hat{x}(t), c(t))$ denote the concatenate state of L-SLS (2), where $c(t)$ is the automaton state of $G_{\mathcal{A}}^M(V, \hat{E})$. Without loss of generality, assume that $G_{\mathcal{A}}(V, E)$ is strong connected and the initial time $t_0 = 0$. The results in the case that a graph consists of several nontrivial SCCs can be obtained similarly.

Proposition 1: (S-procedure [27]) Let $T_0, \dots, T_p \in \mathbb{R}^{n \times n}$ be symmetric matrices. Consider the following two conditions:

S_1 : For all $y \in \mathbb{R}^n \setminus \{0\}$ such that $y^T T_i y \geq 0$, $i = 1, \dots, p$, we have $y^T T_0 y > 0$;

S_2 : There exists $\tau_i \geq 0$, $i = 1, \dots, p$, such that for all $y \in \mathbb{R}^n \setminus \{0\}$,

$$T_0 - \sum_{i=1}^p \tau_i T_i > 0$$

The condition S_2 implies the condition S_1 but not vice versa. It is a nontrivial fact that when $p = 1$, the converse holds. In general, the condition S_2 is more conservative but easier to be checked compared with S_1 . The S-procedure is to verify the validity of S_1 by checking S_2 .

Theorem 1: Consider a labeled digraph $G_{\mathcal{A}}^M(V, \hat{E})$, if there exists Q_1, Q_2, \dots, Q_h and P_1, P_2, \dots, P_h , $h = |V|$, $P_i > 0$, $Q_i > 0$, $i \in \mathbb{N}_h$, and a set of nonnegative scalars $\{\alpha_{(j, \psi, s)}^e\}$ such that for each edge $e = (j, \xi, i) \in \hat{E}$, the following inequalities are satisfied

$$Q_j - \hat{A}_{\xi}^T Q_i \hat{A}_{\xi} > \sum_{(j, \psi, s) \in B(j) \setminus e} \alpha_{(j, \psi, s)}^e (\hat{A}_{\psi}^T P_s \hat{A}_{\psi} - \hat{A}_{\xi}^T P_i \hat{A}_{\xi}) \quad (3)$$

where $B(j)$ denotes the set of edges starting from the vertex j , (j, ψ, s) represents any edge in $B(j) \setminus e$, then the L-SLS (1) is GES under the min-switching control policy

$$e^*(t) = (c(t), l^*(t), c^*(t+1)) = \arg \min_{(c(t), l, c(t+1)) \in B(c(t))} \hat{x}(t)^T \hat{A}_l^T P_i \hat{A}_l \hat{x}(t), t \geq 0 \quad (4)$$

Proof: Suppose that the concatenate state of L-SLS (2) at time t is $(\hat{x}(t), c(t)) = (\hat{x}, j)$, and the next concatenate state after applying the min-switching control policy (4) is $(\hat{x}(t+1), c(t+1)) = (\hat{A}_{\xi} \hat{x}, i)$, as shown in Figure 3, where the labeled edge (j, ξ, i) is obtained by the policy (4).

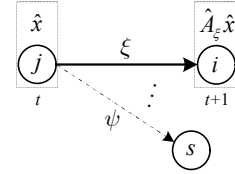


Figure 3. The evolution of the concatenate states

From the control policy (4), we know that for any edge $(j, \psi, s) \in B(j) \setminus e$, there exist

$$\hat{x}(t)^T (\hat{A}_{\psi}^T P_s \hat{A}_{\psi} - \hat{A}_{\xi}^T P_i \hat{A}_{\xi}) \hat{x}(t) \geq 0 \quad (5)$$

Construct the Lyapunov function for the concatenate states at time t as

$$V(\hat{x}(t), c(t)) = \sqrt{\hat{x}(t)^T Q_j \hat{x}(t)} \quad (6)$$

Noticing that by the eq. (5) and the S-procedure, one can see that the eq. (3) implies $\hat{x}(t)^T Q_j \hat{x}(t) > \hat{x}(t)^T \hat{A}_{\xi}^T Q_i \hat{A}_{\xi} \hat{x}(t)$, thus $V(\hat{x}(t), c(t)) > V(\hat{x}(t+1), c(t+1))$. Moreover, there exists a scalar $\varepsilon_3 > 0$ such that, for all $(\hat{x}(t), c(t)) \in \mathbb{R}^n \times V$, $t \geq 0$,

$$V(\hat{x}(t+1), c(t+1)) - V(\hat{x}(t), c(t)) < -\varepsilon_3 \|\hat{x}(t)\|$$

In addition, there exists scalars $0 < \varepsilon_1 \leq \varepsilon_2$ such that,

$$\varepsilon_1 \|\hat{x}(t)\| \leq V(\hat{x}(t), c(t)) \leq \varepsilon_2 \|\hat{x}(t)\|$$

therefore

$$V(\hat{x}(t+1), c(t+1)) < (1 - \varepsilon_3 / \varepsilon_2) V(\hat{x}(t), c(t))$$

which implies

$$V(\hat{x}(t), c(t)) < (1 - \varepsilon_3 / \varepsilon_2)^t V(\hat{x}(0), c(0))$$

the above result yields that:

$$\|\hat{x}_{x(t_0)}^{\sigma(\cdot)}(t)\| < \varepsilon_2 / \varepsilon_1 e^{\ln(1 - \varepsilon_3 / \varepsilon_2)(t - t_0)} \|\hat{x}(t_0)\|, \forall t \geq 0$$

hence, L-SLS (2) is GES, and L-SLS (1) is also GES according to Lemma 1. ■

From Theorem 1, one can observe that: i) each of the Lyapunov Matrices $\{Q_i\}$ and auxiliary matrices $\{P_i\}$ corresponds to the automaton state (i.e. the vertex of the labeled graph); and ii) the inequality condition (3) and the scalar $\alpha_{(j, \psi, s)}^e$ is related with a given edge $e = (j, \xi, i)$ and all the edges issued from the vertex j .

Consider a special case of Theorem 1: $M=1$, $Q_i = P_i$, $i \in \mathbb{N}_h$ and $\sum_{(j,\psi,s) \in B(j)} \alpha_{(j,\psi,s)} = 1$, the stability condition of this special case is given below.

Corollary 2. Consider a labeled digraph $G_{\mathcal{A}}(V, E)$, if there exists Q_1, Q_2, \dots, Q_h , $h=|V|$, $Q_i > 0$, $i \in \mathbb{N}_h$, and a set of nonnegative scalars $\{\alpha_{(j,\psi,s)}\}$ satisfying $\sum_{(j,\psi,s) \in B(j)} \alpha_{(j,\psi,s)} = 1$, such that for each edge $e = (j, \xi, i) \in E$, the following inequalities are satisfied

$$Q_j - A_\xi^T Q_i A_\xi > \sum_{(j,\psi,s) \in B(j)} \alpha_{(j,\psi,s)} (A_\psi^T Q_s A_\psi - A_\xi^T Q_i A_\xi) \quad (7)$$

where the definition of $B(j)$ is same as in Theorem 1, then L-SLS (1) is GES under the min-switching control policy

$$e^*(t) = (c(t), l^*(t), c^*(t+1)) = \arg \min_{(c(t), l, c(t+1)) \in B(c(t))} \hat{x}(t)^T A_l^T P_l A_l \hat{x}(t), t \geq 0 \quad (8)$$

Proposition 2: The sufficient condition in Corollary 2 is equivalent to the Lyapunov-Metzler inequalities condition in theorem 1 proposed in [23].

Proof: The right-hand side of eq. (7) can be rewritten as

$$\sum_{(j,\psi,s) \in B(j)} \alpha_{(j,\psi,s)} (A_\psi^T Q_s A_\psi - A_\xi^T Q_i A_\xi)$$

and further rewrite the above repression as

$$\sum_{(j,\psi,s) \in B(j)} \alpha_{(j,\psi,s)} A_\psi^T Q_s A_\psi - \sum_{(j,\psi,s) \in B(j)} \alpha_{(j,\psi,s)} A_\xi^T Q_i A_\xi$$

since $\sum_{(j,\psi,s) \in B(j)} \alpha_{(j,\psi,s)} = 1$, the above expression is equivalent to

$$\sum_{(j,\psi,s) \in B(j)} \alpha_{(j,\psi,s)} A_\psi^T Q_s A_\psi - A_\xi^T Q_i A_\xi$$

therefore, the eq. (7) is equivalent to

$$Q_j > \sum_{(j,\psi,s) \in B(j)} \alpha_{(j,\psi,s)} A_\psi^T Q_s A_\psi$$

which is equivalent to eq. (8) in [23].

Remark 1: Although Theorem 2 and Corollary 2 are equivalent, the conditions in such two results are generated from different perspectives, i.e., eq. (8) in [23] is listed according to the vertex in $G_{\mathcal{A}}(V, E)$, while eq. (7) in Theorem 2 is listed based on the edge (in other words, the path with $\text{len}(p)=1$).

It should be noted that eq. (3) in Theorem 1 is also listed based on edge. If we follow this step further and consider the case $\text{len}(p) \geq 2$, then a more general results can be obtained, as shown in Theorem 3 where $\text{len}(p)=2$.

Theorem 2: Consider a labeled digraph $G_{\mathcal{A}}^M(V, \hat{E})$, if there exists Q_1, Q_2, \dots, Q_h and P_1, P_2, \dots, P_h , $h=|V|$, $P_i > 0$, $Q_i > 0$, $i \in \mathbb{N}_h$, and two sets of nonnegative scalars $\{\alpha_{(k,\varphi,r)}^{e_{i-1}}\}$ and $\{\alpha_{(j,\psi,s)}^{e_i}\}$ such that for each two-step path $p = (k, \eta, j)(j, \xi, i)$ with $e_{i-1} = (k, \eta, j) \in \hat{E}$, $e_i = (j, \xi, i) \in \hat{E}$, the following inequalities are satisfied

$$\begin{aligned} \hat{A}_\eta^T Q_j \hat{A}_\eta - \hat{A}_\eta^T \hat{A}_\xi^T Q_i \hat{A}_\xi \hat{A}_\eta &> \sum_{(k,\varphi,r) \in B(k) \cap e_{i-1}} \alpha_{(k,\varphi,r)}^{e_{i-1}} (\hat{A}_\varphi^T P_r \hat{A}_\varphi - \hat{A}_\eta^T P_j \hat{A}_\eta) \\ &+ \sum_{(j,\psi,s) \in B(j) \cap e_i} \alpha_{(j,\psi,s)}^{e_i} \hat{A}_\eta^T (\hat{A}_\psi^T P_s \hat{A}_\psi - \hat{A}_\xi^T P_i \hat{A}_\xi) \hat{A}_\eta \end{aligned} \quad (9)$$

where the definition of $B(k)$ and $B(j)$ is same as in Theorem 1, then L-SLS (1) is GES under the min-switching control policy

$$e^*(t) = (c(t), l^*(t), c^*(t+1)) = \arg \min_{(c(t), l, c(t+1)) \in B(c(t))} \hat{x}(t)^T \hat{A}_l^T P_l \hat{A}_l \hat{x}(t), t \geq 0 \quad (10)$$

Proof: Suppose that the concatenate state of L-SLS (2) at time $t-1$ is $(\hat{x}(t-1), c(t-1)) = (\hat{x}, k)$, and the switching edge by applying the min-switching control policy (10) at time $t-1$ is $e_{t-1} = (k, \eta, j)$, then the switching edge by applying the policy (10) at time t is $e_t = (j, \xi, i)$, as shown in Figure 4.

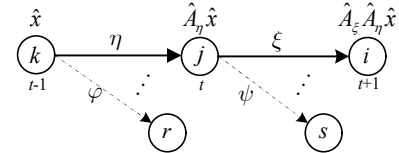


Figure 4. The concatenate states from t-1 to t+1.

It can be seen from (10) that for any edge $(k, \varphi, r) \in B(k) \setminus e_{t-1}$, there exist

$$\hat{x}(t-1)^T (\hat{A}_\varphi^T P_r \hat{A}_\varphi - \hat{A}_\eta^T P_j \hat{A}_\eta) \hat{x}(t-1) \geq 0$$

and for any edge $(j, \psi, s) \in B(j) \setminus e_t$, there exist

$$\hat{x}(t-1)^T \hat{A}_\eta^T (\hat{A}_\psi^T P_s \hat{A}_\psi - \hat{A}_\xi^T P_i \hat{A}_\xi) \hat{A}_\eta \hat{x}(t-1) \geq 0$$

Take the Lyapunov function as (6), it is obvious that

$$\hat{x}(t-1)^T \hat{A}_\eta^T Q_j \hat{A}_\eta \hat{x}(t-1) > \hat{x}(t-1)^T \hat{A}_\eta^T \hat{A}_\xi^T Q_i \hat{A}_\xi \hat{A}_\eta \hat{x}(t-1)$$

which is equivalent to $\hat{x}(t)^T Q_j \hat{x}(t) > \hat{x}(t)^T \hat{A}_\xi^T Q_i \hat{A}_\xi \hat{x}(t)$, thus $V(\hat{x}(t), c(t)) > V(\hat{x}(t+1), c(t+1))$. Therefore, L-SLS (2) is GES, and L-SLS (1) is also GES according to Lemma 1. ■

In Theorem 1 and Theorem 2, each of the auxiliary matrices $\{P_i\}$ is constructed based on the vertex of the labeled digraph. An alternative to construct $\{P_i\}$ can be based on the edge, this leads to:

Theorem 3: Consider a labeled digraph $G_{\mathcal{A}}^M(V, \hat{E})$, if there exists Q_1, Q_2, \dots, Q_h , $h=|V|$, $Q_j > 0$, $j \in \mathbb{N}_h$, multiple sets of positive definite matrices $\{P_{e_i}^{j,k}\} = \{P_{e_1}^{j,k}, P_{e_2}^{j,k}, \dots, P_{e_m}^{j,k}\}$, $k \in \mathbb{N}_n$, $n \geq |B(j)|$, $e_1, e_2, \dots, e_m \in B(j)$, $m = |B(j)|$, $j \in \mathbb{N}_h$, and multiple sets of nonnegative scalars $\{\alpha_{(j,\psi,s)}^{e,k}\}$, $k \in \mathbb{N}_n$, such that for each edge $e = (j, \xi, i) \in \hat{E}$, the following inequalities are satisfied

$$Q_j - \hat{A}_\xi^T Q_i \hat{A}_\xi > \sum_{(j,\psi,s) \in B(j) \cap e} \alpha_{(j,\psi,s)}^{e,k} (\hat{A}_\psi^T P_{e_i}^{j,k} \hat{A}_\psi - \hat{A}_\xi^T P_{e_i}^{j,k} \hat{A}_\xi), k \in \mathbb{N}_n \quad (11)$$

then the L-SLS (1) is GES under the improved min-switching control policy

$$e^* = (c(t), l^*(t), c^*(t+1)) \in \begin{cases} \arg \min_{(c(t), l, c(t+1)) \in B(c(t))} \hat{x}(t)^T \hat{A}_l^T P_l^{j,1} \hat{A}_l \hat{x}(t) \\ \vdots \\ \arg \min_{(c(t), l, c(t+1)) \in B(c(t))} \hat{x}(t)^T \hat{A}_l^T P_l^{j,n} \hat{A}_l \hat{x}(t) \end{cases}, t \geq 0 \quad (12)$$

Proof: Suppose that the concatenate state of L-SLS (2) at time t is $(\hat{x}(t), c(t)) = (\hat{x}, j)$, and for a set of auxiliary matrices $\{P_{e_i}^{j,k}\}$, $k \in \mathbb{N}_n$, the switching edge by applying the

min-switching control policy (12) is $e=(j,\xi,i)$, as shown in Figure 5.

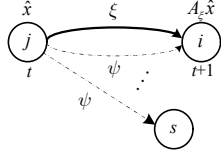


Figure 5. The concatenate states at time t and t+1.

It can be seen from (12) that for all $(j, \psi, s) \in \hat{E} \setminus e$, we have

$$\hat{x}(t)^T (A_\psi^T P_{(j,\psi,s)}^{j,k} A_\psi - A_\xi^T P_{(j,\xi,i)}^{j,k} A_\xi) \hat{x}(t) \geq 0$$

Take the Lyapunov function as the same form of (6), it is obvious that $\hat{x}(t)^T Q_j \hat{x}(t) > \hat{x}(t)^T \hat{A}_\xi^T Q_i \hat{A}_\xi \hat{x}(t)$, thus $V(\hat{x}(t), c(t)) > V(\hat{x}(t+1), c(t+1))$. Therefore, L-SLS (2) is GES, and L-SLS (1) is also GES according to Lemma 1. ■

Remark 2: The min-switching control policy in Theorem 1 and Theorem 2 outcomes only one edge, while the improved policy in Theorem 3 outcomes a set of edges which includes no less than one edge. This implies that we have more flexibility in the design of control strategy to meet the control requirement besides stability.

Notice that each set of auxiliary matrices $\{P_e^{j,k}\}$, $k \in \mathbb{N}_n$, in Theorem 3 is solved independently. Therefore, the outcomes of the improved policy (12) cannot be guaranteed to be different from each other. This means some sets of the auxiliary matrices may be redundant. In the following, we present a more efficient algorithm to compute $\{P_e^{j,k}\}$ sequentially.

Algorithm 1: a sequential solving algorithm for $\{P_e^{j,k}\}$

Initialization: given the finite state automaton $G_{\mathcal{A}}^M(V, \hat{E})$;

Solving $\{Q_j\}$ and $\{P_e^{j,1}\}$, $j \in \mathbb{N}_n$: for $e=(j, \xi, i) \in \hat{E}$, solving eq. (11) where $k=1$;

Solving $\{P_e^{j,k}\}$, $j \in \mathbb{N}_n, k \geq 2$:

for any $j \in \mathbb{N}_n$:

number the edges $e=(j, \xi, i) \in B(j)$ from 1 to $|B(j)|$;
 $k=1$;

for any $e=(j, \xi, i) \in B(j)$:

step 1: for the k -th edge $e'=(j, \eta, s) \in B(j) \setminus e$, solve eq. (11) and the following eq. (13):

$$\sum_{(j,\psi,s) \in B(j) \setminus e'} \beta_{(j,\psi,s)}^{e,k} (\hat{A}_\psi^T P_{(j,\psi,s)}^{j,k} \hat{A}_\psi - \hat{A}_\eta^T P_{(j,\eta,s)}^{j,k} \hat{A}_\eta) \geq \sum_{(j,\psi,s) \in B(j) \setminus e} \beta_{(j,\psi,s)}^{e,1} (\hat{A}_\psi^T P_{(j,\psi,s)}^{j,1} \hat{A}_\psi - \hat{A}_\xi^T P_{(j,\xi,i)}^{j,1} \hat{A}_\xi) \quad (13)$$

if there is a solution for the k -th edge, then we can obtain the k -th set of auxiliary matrices $\{P_e^{j,k}\}$;

step 2: if eq. (11) and eq. (13) have a solution in step 1, then recode $\{P_e^{j,k}\}$ and the corresponding edges;

step 3: $k=k+1$;

end

end

IV. NUMERICAL EXAMPLE

Consider an L-SLS (1) with language constraint as in Figure 6. The initial system state, and automaton state are $x_0 = [2 \ -1]^T$ and v_1 respectively. System matrices are

$$A_1 = \begin{bmatrix} -0.6 & 0 \\ -0.1 & 0.4 \end{bmatrix}, A_2 = \begin{bmatrix} 0.7 & 0 \\ 0 & 0.4 \end{bmatrix}, A_3 = \begin{bmatrix} 0.4 & -0.1 \\ 0 & 0.8 \end{bmatrix}$$

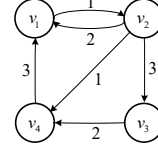


Figure 6. The automaton of the numerical example.

By Theorem 4, we can solve $\{Q_j\}$ and three sets of $\{P_e^{j,k}\}$ as following:

$$\{Q_j\}: Q_1 = \begin{bmatrix} 1.6477 & -0.0047 \\ -0.0047 & 1.7971 \end{bmatrix}, Q_2 = \begin{bmatrix} 1.7415 & -0.0419 \\ -0.0419 & 1.7948 \end{bmatrix}$$

$$Q_3 = \begin{bmatrix} 1.6995 & 0.0017 \\ 0.0017 & 1.4711 \end{bmatrix}, Q_4 = \begin{bmatrix} 1.3826 & -0.0378 \\ -0.0378 & 1.7970 \end{bmatrix}$$

$$\{P_e^{j,1}\}: \begin{cases} P_{(v_2,2,v_1)}^{v_2,1} = \begin{bmatrix} 1.2590 & -0.0027 \\ -0.0027 & 1.3924 \end{bmatrix} \\ P_{(v_2,3,v_3)}^{v_2,1} = \begin{bmatrix} 1.3577 & 0.0181 \\ 0.0181 & 1.2330 \end{bmatrix} \\ P_{(v_2,1,v_4)}^{v_2,1} = \begin{bmatrix} 1.3782 & 0.0041 \\ 0.0041 & 1.3436 \end{bmatrix} \end{cases}$$

$$\{P_e^{j,2}\}: \begin{cases} P_{(v_2,2,v_1)}^{v_2,2} = \begin{bmatrix} 2.4032 & 0.0031 \\ 0.0031 & 1.8477 \end{bmatrix} \\ P_{(v_2,3,v_3)}^{v_2,2} = \begin{bmatrix} 2.5005 & 0.0319 \\ 0.0319 & 2.1224 \end{bmatrix} \\ P_{(v_2,1,v_4)}^{v_2,2} = \begin{bmatrix} 2.4537 & -0.0113 \\ -0.0113 & 2.5516 \end{bmatrix} \end{cases}, \{P_e^{j,3}\}: \begin{cases} P_{(v_2,2,v_1)}^{v_2,3} = \begin{bmatrix} 2.1423 & 0.0046 \\ 0.0046 & 1.9645 \end{bmatrix} \\ P_{(v_2,3,v_3)}^{v_2,3} = \begin{bmatrix} 2.4124 & -0.0188 \\ -0.0188 & 2.5493 \end{bmatrix} \\ P_{(v_2,1,v_4)}^{v_2,3} = \begin{bmatrix} 2.3920 & -0.0032 \\ -0.0032 & 2.4281 \end{bmatrix} \end{cases}$$

By applying the improved min-switching control policy (12), the resulting trajectory of system state is shown in Figure 7. The switching sequence and automaton state sequence is shown in Figure 8 and Figure 9 respectively.

In Figure 8 and Figure 9, one can see that there are multiple choices in label and automaton state at some moments.

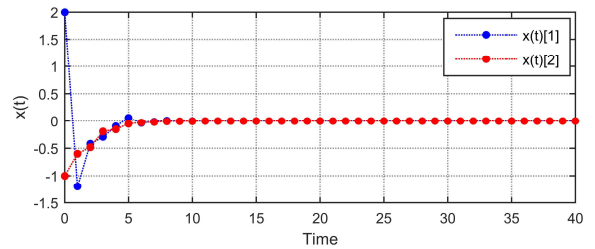


Figure 7. System state $x(t)$.

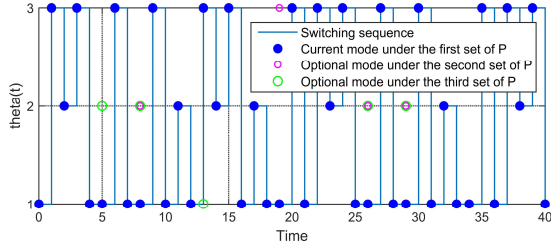


Figure 8. Switching sequence $\sigma(t)$.

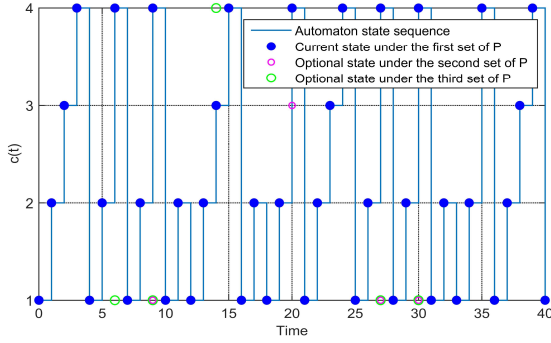


Figure 9. Automaton state $c(t)$.

In comparison, we use sequential solving Algorithm 1 to solve the problem again. Take the vertex v_2 as an example, and assume the current outcome of the min-switching control policy is $e = (v_2, 1, v_4)$. Take $\{P_e^{v_2,1}\}$ in above as the first set of auxiliary matrices, then the other two sets of auxiliary matrices are obtained as below. This implies that the any of the two edges $e = (v_2, 2, v_1)$ and $e = (v_2, 3, v_3)$ can be chosen as the next switching path, as shown in Figure 10(a).

$$\{P_e^{v_2,2}\} : \begin{cases} P_{(v_2,2,v_1)}^{v_2,2} = \begin{bmatrix} 1.5136 & -0.0873 \\ -0.0873 & 1.6983 \end{bmatrix} \\ P_{(v_2,3,v_3)}^{v_2,2} = \begin{bmatrix} 2.4052 & 0.0012 \\ 0.0012 & 2.9794 \end{bmatrix} \\ P_{(v_2,1,v_4)}^{v_2,2} = \begin{bmatrix} 2.7006 & 0.1151 \\ 0.1151 & 1.9852 \end{bmatrix} \end{cases} \quad \{P_e^{v_2,3}\} : \begin{cases} P_{(v_2,2,v_1)}^{v_2,3} = \begin{bmatrix} 3.2952 & 0.2153 \\ 0.2153 & 0.8012 \end{bmatrix} \\ P_{(v_2,3,v_3)}^{v_2,3} = \begin{bmatrix} 1.4597 & 0.1043 \\ 0.1043 & 0.0615 \end{bmatrix} \\ P_{(v_2,1,v_4)}^{v_2,3} = \begin{bmatrix} 1.7445 & -0.6129 \\ -0.6129 & 4.0683 \end{bmatrix} \end{cases}$$

Similarly, if the current outcome of the min-switching control policy is $e = (v_2, 3, v_3)$. Then the two sets of auxiliary matrices can be obtained. Therefore, the two edges $e = (v_2, 2, v_1)$ and $e = (v_2, 1, v_4)$ are optional as the next switching path, as shown in Figure 10(b).

$$\{P_e^{v_2,2}\} : \begin{cases} P_{(v_2,2,v_1)}^{v_2,2} = \begin{bmatrix} 1.5913 & -0.1587 \\ -0.1587 & 2.4309 \end{bmatrix} \\ P_{(v_2,3,v_3)}^{v_2,2} = \begin{bmatrix} 3.4357 & 0.2284 \\ 0.2284 & 1.7123 \end{bmatrix} \\ P_{(v_2,1,v_4)}^{v_2,2} = \begin{bmatrix} 0.1251 & -0.2020 \\ -0.2020 & 1.5844 \end{bmatrix} \end{cases} \quad \{P_e^{v_2,3}\} : \begin{cases} P_{(v_2,2,v_1)}^{v_2,3} = \begin{bmatrix} 0.3416 & -0.1504 \\ -0.1504 & 2.9452 \end{bmatrix} \\ P_{(v_2,3,v_3)}^{v_2,3} = \begin{bmatrix} 3.0898 & 0.1675 \\ 0.1675 & 1.6929 \end{bmatrix} \\ P_{(v_2,1,v_4)}^{v_2,3} = \begin{bmatrix} 2.9825 & 0.3555 \\ 0.3555 & 1.5893 \end{bmatrix} \end{cases}$$

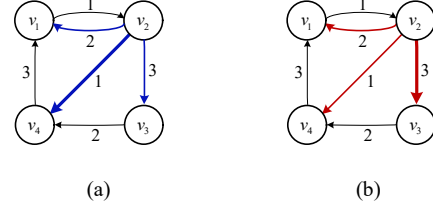


Figure 10. The automaton of the numerical example.

V. CONCLUSION

This paper deals with the globally exponentially stabilizable of switched system with language constraint which is generated by a non-deterministic finite state automaton. Based on the S-procedure characterization, sufficient conditions of GES are proposed for the language constrained switched system. We prove that the condition can be degenerated to the exiting Lyapunov-Metzler condition. In addition, a more general condition is given by revising the construction of candidate auxiliary matrices and the min-switching control policy. Finally, a numerical example is given to demonstrate the proposed results.

REFERENCES

- [1] X. Wu, K. Zhang, M. Cheng, "Computational method for optimal machine scheduling problem with maintenance and production," *International Journal of Production Research*, vol. 55, no. 6, pp. 1791–1814, 2017.
- [2] P. Prabhakar, S. Mitra, M. Viswanathan, "On convergence of concurrent systems under regular interactions," in *concur 2009 - concurrency theory, proceedings*, vol. 5710, pp. 527–541, 2009.
- [3] D. Ma, J. Zhao, "Stabilization of networked switched linear systems: An asynchronous switching delay system approach," *Systems & Control Letters*, vol. 77, pp. 46–54, Mar. 2015.
- [4] D. Liberzon, A.S. Morse, "Global finite-time stabilization of a class of switched nonlinear systems with the powers of positive odd rational numbers," *Automatica*, vol. 54, pp. 360–373, Apr. 2015.
- [5] H. Lin, P.J. Antsaklis, "Stability and stabilizability of switched Linear systems: a survey of recent results," *IEEE Transactions on Automatic Control*, vol. 54, no. 2, pp. 308–322, Feb. 2009.
- [6] R. Jungers, *The Joint Spectral Radius*. Springer, Berlin, Heidelberg, 2009, pp. 38–43.
- [7] A.A. Ahmadi, R.M. Jungers, P. A. Parrilo, M. Roozbehani, "Joint spectral radius and path-complete graph Lyapunov functions," *SIAM Journal on Control and Optimization*, vol. 52, no. 1, pp. 687–717, 2014.
- [8] R.M. Jungers, A.A. Ahmadi, P.A. Parrilo, M. Roozbehani, "A Characterization of Lyapunov Inequalities for Stability of Switched Systems," *IEEE Transactions on Automatic Control*, vol. 62, no. 6, June 2017.
- [9] D. Angeli, N. Athanasopoulos, R.M. Jungers, M. Philippe, "Path-complete graphs and common Lyapunov functions," *Proceedings of the 20th International Conference on Hybrid Systems: Computation and Control*, Apr. 2017.
- [10] D. Lee, J. Hu, "On the graph control Lyapunov function for stabilization of discrete-time switched linear systems," *2017 American Control Conference (ACC)*, pp. 5159–5164, 2017.
- [11] C. Briat, "Convex lifted conditions for robust l_2 -stability analysis and l_2 -stabilization of linear discrete-time switched systems with minimum dwell-time constraint," *Automatica*, vol. 50, no. 3, pp. 976–983, 2014.
- [12] W. Xiang, "Necessary and sufficient condition for stability of switched uncertain linear systems under dwell-time constraint," *IEEE Transactions on Automatic Control*, vol. 61, no. 11, pp. 3619–3624, Nov. 2016.
- [13] J. C. Geromel, P. Colaneri, "Stability and stabilization of discrete time switched systems," *International Journal of Control*, vol. 79, no. 7, pp. 719–728, Feb. 2006.

- [14] M. Fiacchini, A. Girard, M. Jungers, "On the stabilizability of discrete-time switched linear systems: novel conditions and comparisons," *IEEE Transactions on Automatic Control*, vol. 61, no. 5, pp. 1181-1193, May, 2016.
- [15] N. Chomsky, *Syntactic structures*. Germany: Mouton & Co, February, 1957.
- [16] N. Chomsky, "On certain formal properties of grammars," *Information and Control*, vol. 2, no. 2, pp. 137-167, 1959.
- [17] J. Kari, "Synchronizing finite automata on Eulerian digraphs," *Theoretical Computer Science*, vol. 295, no. 1-3, pp. 223-232, 24 Feb. 2003.
- [18] X. Dai, "A gel'fand-type spectral radius formula and stability of linear constrained switching systems," *Linear Algebra and its Applications*, vol. 436, no. 5, pp. 1099-1113, 2012.
- [19] M. Philippe, R. Essick, G.E. Dullerud, R.M. Jungers, "Stability of discrete-time switching systems with constrained switching sequences," *Automatica*, vol. 72, pp. 242-250, Oct. 2016.
- [20] B. Legat, P.A. Parrilo, R.M. Jungers, "An entropy-based bound for the computational complexity of a switched system," *IEEE Transactions on Automatic Control (Early Access)*, 04 Mar. 2019.
- [21] Y. Wang, N. Roohi, G.E. Dullerud, M. Viswanathan, "Stability Analysis of Switched Linear Systems Defined by Regular Languages," *IEEE Transactions on Automatic Control*, vol. 62, no. 5, May 2017.
- [22] M. Fiacchini, M. Jungers, A. Girard, "Stabilization and control Lyapunov functions for language constrained discrete-time switched linear systems," *Automatica*, vol. 93, pp. 64-74, July 2018.
- [23] M. Jungers, A. Girard, M. Fiacchini, "Language constrained stabilization of discrete-time switched linear systems: a Lyapunov-Metzler inequalities approach," *2016 IEEE 55th Conference on Decision and Control (CDC)*, pp. 5539-5544, 2016.
- [24] W.P.M.H. Heemels, A. Kundu, J. Daafouz, "On Lyapunov-Metzler inequalities and S-procedure characterizations for the stabilization of switched linear systems," *IEEE Transactions on Automatic Control*, vol. 62, no. 9, pp. 4593-4597, Sep. 2017.
- [25] F. Bullo, *Lectures on Network Systems*, private communication, 30 June 2016.
- [26] W. Xiang, H.D. Tran, T.T. Johnson, "Robust exponential stability and disturbance attenuation for discrete-time switched systems under arbitrary switching," *IEEE Transactions on Automatic Control*, vol. 63, no. 5, pp. 1450-1456, May 2018.
- [27] E.W. Dijkstra. *A Discipline of Programming*. Prentice Hall, 1976.
- [28] S. Boyd, L. El Ghaoui, E. Feron, V. Balakrishnan, "Linear matrix inequalities in system and control theory," *Proceedings of the IEEE*, vol. 86, no. 12, Dec. 1998.